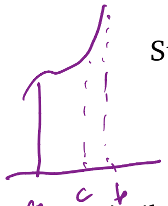
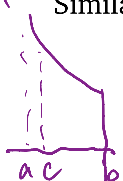
Definition 5.2 (Improper integrals of Type 2)**.** The improper integrals defined in Definition 5.1 has infinite intervals of integration, but the values of the integrand are finite on the intervals of the integration. We also generalize definite integrals where the integrand may go to $\pm\infty$ over the interval of integration.



Suppose that $f(x)$ is continuous on (a, b), but $\lim_{x\to b^-} f(x) = \pm\infty$. Then we define:

$$\int_a^b f(x)dx := \lim_{c\to b^-}\int_a^c f(x)dx.$$

Similarly, if $\lim_{x\to a^+} f(x) = \pm\infty$,



$$\int_a^b f(x)dx := \lim_{c\to a^+}\int_c^b f(x)dx.$$

$$\int_c^1 \frac{1}{x^p}dx = \begin{cases} \left.\frac{x^{1-p}}{1-p}\right|_c^1 = \frac{1-c^{1-p}}{1-p} & p\neq 1 \\ \ln|x|\Big|_c^1 = -\ln c & p=1 \end{cases}$$

Example 5.6. 1. $\int_0^1 \frac{1}{x^p}dx$ $= \lim_{c\to 0^+}\int_c^1 \frac{1}{x^p}dx = \begin{cases} \frac{1}{1-p} & \text{when } p<1 \\ +\infty & \text{when } p\geq 1 \end{cases}$

2. $\int_0^1 \frac{\cancel{1}}{\ln x}dx$

3. (mixed type) $\int_{-\infty}^1 \frac{1}{x^3}dx$

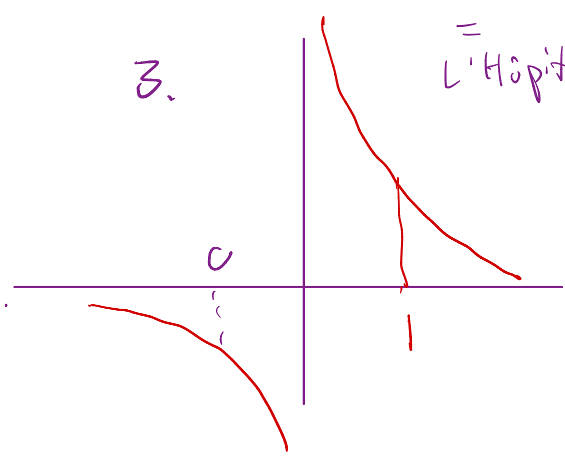
2. $\int_0^1 \ln x\, dx = \lim_{c\to 0^+}\int_c^1 \ln x\, dx$

$\int_c^1 \ln x\, dx = x\ln x\Big|_c^1 - \int_c^1 x\cdot\frac{1}{x}dx$
$= -c\ln c - (1-c)$

$= \lim_{c\to 0^+} \frac{\ln(\frac{1}{c})}{\frac{1}{c}} - 1 = \lim_{u\to+\infty}\frac{\ln u}{u} - 1$

$\underset{\text{L'Hôpital}}{=} \lim_{x\to+\infty}\frac{\frac{1}{u}}{1} - 1 = -1$

3.



$\int_{-\infty}^1 \frac{1}{x^3}dx = \underbrace{\overset{\text{(conv)}}{\int_{-\infty}^c \frac{1}{x^3}dx}}_{\text{Type I}} + \underbrace{\int_c^0 \frac{1}{x^3}dx}_{\text{Type 2 (div)}}$

$+ \underbrace{\int_0^1 \frac{1}{x^3}dx}_{\text{Type 2. (div)}}$

defined

is divergent.

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Chapter 11: Ordinary Differential Equations

Learning Objectives:
(1) Solve first-order linear differential equations and initial value problems.
(2) Explore analysis with applications to dilution models.

1 Ordinary Differential Equations

Definition 1.1. An ordinary differential equation (ODE) is an equation involving one or more derivatives of an unknown function $y(x)$ of 1-variable. A differential equation for a multi-variable function is called a "partial differential equation" (PDE).

The order of an ordinary differential equation is the order of the highest derivative that it contains.

Example 1.1.

DIFFERENTIAL EQUATION	ORDER
1. $\frac{dy}{dx} + 0 \cdot y = 4x$ linear.	1
$\frac{d^3y}{dt^3} - t\frac{dy}{dt} + t(y-1) = e^t$ linear	3
1. $y' + 1 \cdot y = 2x^2$ linear. ↑ inhomogeneous term	1

1. $y^{(3)} + 0 \cdot y^{(2)} + (-t)\,y' + (t)\,y = t + e^t$

Example 1.2. 1. $y\,y'' + e^y = x^2 \ln y'$ is a second order ODE. ← nonlinear

2. $f_2(x)y'' + f_1(x)y' + f_0(x)y = g(x)$, $f_2(x) \neq 0$. This is a second order *linear* ODE in the function $y(x)$. $g(x)$ is called the *inhomogeneous term*; the left hand side of the equation is called the *homogeneous part* of the this linear ODE; $f_2(x)y" + f_1(x)y' + f_0(x)y = 0$ is called the associated homogeneous linear ODE of the linear ODE given above. A linear ODE with inhomogeous term 0 is called a *homogeneous* linear ODE.

3. The ODE in 1. is non-linear. The second ODE in Example 1.1 is linear with inhomogeneous term e^t.

General n-th order linear ODE

$$\underbrace{f_n(x)\,y^{(n)} + f_{n-1}(x)\,y^{(n-1)} + \cdots + f_1(x)\,y' + f_0(x)\,y}_{\text{homogeneous part}} = g(x) \rightarrow \text{inhomogeneous part}$$

$x_1, x_2, \cdots x_n$ variables. → a linear combination of $x_1, \cdots x_n$

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

Remark. $\sum_{i=1}^{n} a_i x_i = b$, where a_i, b are constants ("coefficients") is said to be a linear equation in the variables $x_1, \ldots, x_n$. b is called the inhomogeneous term, and the equation is said to be homogeneous when $b = 0$. For differential equations, functions of x play the roles of "coefficients" $a_1, \ldots, a_n$, b, and $y^{(i)}$, $i = 0, 1, \ldots$ play the roles of "variables".

Definition 1.2. A function $y = y(x)$ is a **solution** of an ordinary differential equation on an open interval if the equation is satisfied identically on the interval when y and its derivatives are substituted into the equation.

for all x

Remark. The solution might not exist; it might not be unique.

in fact $y(x) = Ce^{2x}$ is a solution for any constant C

Example 1.3. $y(x) = e^{2x}$ is a solution to the ODE $y'' - 4y' + 4y = 0$. $y(x) = 4e^{2x}$ is another solution.

$(e^{2x})'' - 4(e^{2x})' + 4e^{2x} = 4e^{2x} - 4 \cdot 2 e^{2x} + 4e^{2x} = 0$

Example 1.4. Find the solution of $\frac{d}{dx}y = 4x$, or equivalently, $y'(x) = 4x$.

Solution. Integrate both sides: $y(x) = \displaystyle\int 4x\, dx = 2x^2 + C$, where C is an arbitrary constant.

Then, $y = 2x^2 + C, \quad C \in \mathbb{R}$ is called general solution of $y'(x) = 4x$.

Choose any C, e.g. $C = 5$, we get a particular solution $y = 2x^2 + 5$. ■

For a first-order equation, the single arbitrary constant can be determined by specifying the value of the unknown function $y(x)$ at an arbitrary x-value x_0, say $y(x_0) = y_0$. This is called an initial condition, and the problem of solving a first-order equation subject to an initial condition is called a **first-order initial-value problem**.

Example 1.5.

$$\begin{cases} y'(x) = 4x \\ y(5) = 20 \end{cases}$$

is an initial value problem.

General solution $y = 2x^2 + C$ should satisfy the initial condition $y(5) = 20$, i.e.

$$20 = 2(5)^2 + C \quad \Rightarrow \quad C = -30.$$

So, the unique solution to the initial value problem is $y = 2x^2 - 30$.

Remark. We saw that the general solution to a first order ODE typically involves an indeterminate constant C. More generally, the general solution to an n-th order ODE typically involves n indeterminate constants. An initial value problem for an n-th order ODE thus has n initial conditions, often of the form $y^k(x_0) = a_k$, $k = 0, 2, \ldots, n-1$, where x_0 and a_k are constants.

Solving a general ODE is typically very difficult, and there is no general algorithm for doing so. We shall discuss only some particularly simple cases.